BEM stiffness-like matrices compression for coupling with FEM in Seismic Wave Propagation

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Seismic waves are waves that travel through the earth, for example, as a result of an earthquake or an explosion. In Seismic Engineering we are interested in predict how this waves could affect buildings in order to optimize their design.[1]

Since seismic sources are commonly far away their are treated as plane before the incidence with obstacles, called scatterers.
Problem - continued

So we know the incident wave and the total displacement field could be written as

$$u = u^i + u^s$$

Where the superscript $i$ refers to incident variables and $s$ to the scattered wave.
Differential equations

When we assume an Elastic behavior we fell into the Navier-Cauchy equation

\[(G + \lambda)u_{j,ij} + Gu_{i,jj} + f_i = \rho \dddot{u}_i.\]

Or in vector notation

\[(G + \lambda) \nabla(\nabla \cdot u) + G \nabla^2 u + f = \rho \frac{\partial^2 u}{\partial t^2}.\]
Numerical Methods

Analytical solutions for the wave scattering problems could be find just for very particular cases (simple geometries, 1D or 2D simplifications, etc.) So, it’s mandatory to suggest numerical alternatives to the problem. Domain decomposition techniques are widely used for engineering problems, e.g.

• Finite Element Methods (FEM),
• Finite Difference Methods (FDM).

There exists another kind of methods termed boundary methods where the dimensionality of the problem is reduced in one.

• Direct Boundary Element Method (DBEM),
• Indirect Boundary Element Method (IBEM).
We are interested in a hybrid method between a Finite Element Method and a Boundary Element Method, the main idea of coupling these methods is to get better results that those obtained for each method itself.

While problems with unbounded domains are accurately described with the BEM, highly heterogeneous domains (e.g., scatterers) are easily modeled with FEM. The main disadvantage of this hybrid approach however, is the fact that the matrices resulting from the BEM contribution are fully populated and thus computationally demanding.
In the Direct case, and assuming a smooth surface, is

\[
\frac{1}{2} u_i(\xi) \delta_{ij} = \int_S \left[ G_{ij}(x, \xi)t_i(x) - H_{ij}(x, n_x; \xi)t_i(x) \right] dS(x),
\]

in the indirect case are

\[
u_i(x) = \int_S G_{ij}(x, \xi) \phi_j(\xi) dS(\xi),
\]

\[
t_i(x) = \frac{1}{2} \phi_j(x) \delta_{ij} + \int_S H_{ij}(x; \xi, n_\xi) \phi_j(\xi) dS(\xi).
\]
Indirect and Direct formulations Equivalence

The Direct BIE yields to a system of equations

\[
[\mathbf{H}]_{DBEM} \{\mathbf{u}\} = [\mathbf{G}] \{\mathbf{t}\}.
\]

The Indirect BIE yields to the systems

\[
\{\mathbf{u}\} = [\mathbf{G}] \{\mathbf{\phi}\},
\]

\[
\{\mathbf{t}\} = [\mathbf{H}]_{IBEM} \{\mathbf{\phi}\}.
\]

And we get

\[
[\mathbf{H}]_{IBEM} [\mathbf{G}]^{-1} = [\mathbf{G}]^{-1} [\mathbf{H}]_{DBEM},
\]

\[
[\mathbf{H}]_{IBEM} = [\mathbf{G}]^{-1} [\mathbf{H}]_{DBEM} [\mathbf{G}].
\]
BEM/FEM Coupling

When a problem is tackled with the FEM one gets a system of equations with the form

\[ [K] \{u\} = \{F\}, \]

K is the stiffness matrix, F the external forces and u the displacements (that are commonly what one want to know).

In order to make the coupling two conditions are imposed between subdomains:

• Compatibility condition (displacements must be continuous),
• Equilibrium condition.
From the discrete versions of BEM we want to isolate the tractions to get stiffness-like matrices and couple both methods, and we get

\[
[K]_{\text{DBEM}} = [G]^{-1} [H]_{\text{DBEM}},
\]

\[
[K]_{\text{IBEM}} = [H]_{\text{IBEM}} [G]^{-1}.
\]

So

\[
[K]_{\text{IBEM}} = [G]^{-1} [H]_{\text{DBEM}} = [K]_{\text{DBEM}}.
\]
Goal

Boundary Elements stiffness-like matrices are fully populated and thus computationally demanding. In order to maintain a computationally competitive algorithm we enforced a banded condition over the resultant stiffness-like matrices naturally introducing loss of accuracy into the solution but increasing the computation capability like for instance allowing the use of special solvers. In this work we studied this loss of accuracy with regards to two “banding” criteria.
Banding Criteria

Two banding criteria were established

• Threshold (Herod’s Method), first the maximum value of the matrix is found and then all the values below a percentage of this number, fixed beforehand, are replaced with zeros.

• Half-bandwidth, a half-bandwidth value is fixed beforehand and all the matrix entries out of this band are replaced with zeros.
Results – Thresholded Matrices

Thr = 0.01  Thr = 0.001  Thr = 0.0001
Results – Threshold criterion

P-wave vertically incident U-Field

P-wave vertically incident V-Field
Results – Threshold criterion

\[
\begin{align*}
\text{Thr} &= 0 \\
\text{Thr} &= 0.00001 \\
\text{Thr} &= 0.0001 \\
\text{Thr} &= 0.01
\end{align*}
\]
Results – Threshold criterion

\[ \text{Thr} = 0 \]

\[ \text{Thr} = 0.0001 \]

\[ \text{Thr} = 0.00001 \]

\[ \text{Thr} = 0.01 \]
Results – Threshold criterion
Results – Threshold criterion

Thr = 0

Thr = 0.00001

Thr = 0.0001

Thr = 0.01

SV 30° U Field
Results – Threshold criterion
Results – bandwidth criterion
Results – bandwidth criterion

- $H_{bw} = 1$
- $H_{bw} = 0.26$
- $H_{bw} = 0.13$
- $H_{bw} = 0.03$
Results – bandwidth criterion

$H_{bw} = 1$

$H_{bw} = 0.26$

$H_{bw} = 0.13$

$H_{bw} = 0.03$
Results – bandwidth criterion
Results – bandwidth criterion

Hbw = 1
Hbw = 0.25
Hbw = 0.09
Hbw = 0.02
Results – bandwidth criterion

Hbw = 1

Hbw = 0.25

Hbw = 0.09

Hbw = 0.02
Results – bandwidth criterion – matrices properties
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Conclusions

• We coupled FEM and BEM algorithms and obtained results for problems in Wave Propagation in unbounded domains.

• Furthermore we banded the resulting BEM stiffness-like matrices to cast them in the form of standard FEM matrices.

• We computed norms (and errors) for each banded matrix in order to obtain an estimator that could be used as criteria to establish a half-bandwidth size without getting a high error in the results.
Aknowledgement

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References


References


Thanks!!

Questions?